Introduction to planar maps – exercise session

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Guillaume Chapuy
Université Paris Cité and CNRS
guillaume.chapuy@irif.fr

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Exercise 0 – Crash course on Catalan numbers (optional)

Let a_n be the number of rooted plane trees with n edges (=rooted planar maps without cycles, =rooted planar maps with a unique face) and let $A(z) = \sum_{n\geq 0} a_n z^n$.

1. Convince yourself of the following: 1/ removing the root edge in a nontrivial rooted plane tree, I naturally obtain two rooted plane trees, and I know how to go back. 2/ this is implies that $A(z) = 1 + zA(z)^2$. If this is not clear to you, no problem, read an introductory course on generating functions and combinatorial classes (e.g.: Flajolet-Sedgewick).

2. Deduce that $a_n = \mathrm{Cat}(n) := \frac{1}{n+1}\binom{2n}{n}$. (Best way to do it: apply the Lagrange inversion formula to $B(z) := A(z) - 1 = z(1 + B(z))^2$, see the footnote a bit further down).

3. Give a least one bijective proof of the Catalan number formula[1].

4. (Experts) How many (truly) different proofs of the Catalan number formula do you know?

Exercise 1 – Let's draw

1. Draw the 2 (resp. 9) rooted planar maps with 1 (resp. 2) edges, without looking at the course notes! If you feel like it, also do the 54 maps with 3 edges (I recommend you draw each unrooted map only once, and just indicate its inequivalent rootings).

Exercise 2 – Mullin's nice formula for tree-rooted maps.

A tree-rooted map is a rooted map equipped with a distinguished spanning tree. We recall that the number of rooted plane trees (rooted planar maps with one-face, or equivalently rooted planar maps whose underlying graph is a tree) with m edges is the Catalan number $\mathrm{Cat}(m) = \frac{1}{m+1}\binom{2m}{m}$.

1. Let M be a planar map with a spanning tree T. Show that the set of all dual-edges of edges not in T form a spanning tree of the dual map M^*.

2. Deduce the Euler formula: $v + f = e + 2$ (we will not use it, but it's a nice consequence!).

3. Convince yourself (draw a picture!) that when you follow the tour of T, you simultaneously follow the tour of T^*. Given two rooted plane trees T and T^* with a total of n edges, what sort of extra information do you need to glue them together and form a map M?

[1] most classical one: consider Dyck paths instead of rooted plane trees. A Dyck path is the list of the height of corners when you do the tour of the tree, it is a path with ± 1 steps, of length $2n$ going from 0 to 0 and staying nonnegative. A Dyck path with an extra -1 step is called a Lukaciewicz word. Now prove the cycle lemma: given any path with $(n+1)$ steps -1 and n steps $+1$ (but possibly being negative at some places), exactly one of its $(2n+1)$ cyclic conjugates is a Lukaciewicz word. This proves that $a_n = \frac{1}{2n+1}\binom{2n+1}{n}$.

4. Show that the number of tree-rooted planar maps with n edges is equal to

$$\sum_{k=0}^{n} \binom{2n}{2k} \mathrm{Cat}(k)\mathrm{Cat}(n-k) = \mathrm{Cat}(n)\mathrm{Cat}(n+1).$$

5. Bonus: Go read Olivier Bernardi's paper (or find some nice slides) for a (highly non trivial)sum-free bijective proof of the last result!

6. Bonus 2: what is the dual of a spanning tree for a map drawn on a surface of higher genus? say, a torus??? Note: For maps on higher genus surfaces we require that the faces are simply connected.

Exercise 3 – Have fun with the BMJ algorithm, counting all sorts of maps.

In the lectures (see the notes!) we have seen how to write the Tutte equation counting maps, and how to solve it. Given an equation of the form

$$E(A(z,u), a(z), z, u) = 0$$

with unknown series $A(z,u), a(z)$ the BMJ algorithm looks for a series $U = U(z)$ such that

$$(\partial_1 E)(A(z,U(z)), a(z), z, U(z)) = 0,$$

which by the chain rule also implies that

$$(\partial_4 E)(A(z,U(z)), a(z), z, U(z)) = 0.$$

We then perform algebraic elimination to find polynomial equations for $A(z,U(z)), U(z)$... and $a(z)$!

In this exercise we ask you to write the analogue of the Tutte equation for different families of planar maps (in each case you have to find what is the good choice of "catalytic" variable). You can then solve it using BMJ on your favourite computer, and deduce the corresponding numbers. As in the lecture we write $\Delta(A(z,u)) := \frac{A(z,u)-A(z,1)}{u-1}$, and we recall that $\Delta u^k = u^{k-1} + u^{k-2} + \cdots + 1$ which is useful to "add a diagonal in a root face in all possible ways".

1. Rooted maps with n edges (done in class)

$$F(z,u) = 1 + zu^2 F(z,u)^2 + zu\Delta(uF(z,u)) \quad , \quad f_n = \frac{2 \cdot 3^n}{n+2}\mathrm{Cat}(n).$$

2. Rooted bipartite maps with n edges.

$$F(z,u) = 1 + zuF(z,u)^2 + zu\Delta F(z,u) \quad , \quad f_n = \frac{3 \cdot 2^{n-1}(2n)!}{n!(n+2)!}.$$

3. Non-separable maps with n edges (i.e. maps without a cut-vertex, i.e. maps with a 2-connected graph; in particular we forbid loops)

$$F(z,u) = 1 + zuF(z,u)(1 + \Delta F(z,u)) \quad , \quad f_n = \frac{2(3n)!}{(n+1)!(2n+2)!}.$$

4. Non-separable and *cubic* maps (all vertices of degree 3) with $n+1$ faces.

$$F(z,u) = u + zu^2 F(z,u)\Delta F(z,u) \quad , \quad f_n = \frac{2^n(3n)!}{(n+1)!(2n+2)!}.$$

5. Three-connected cubic maps with $n+1$ faces.

$$F(z,u) = 1 + zuF(z,u)\Delta F(z,u) \quad , \quad f_n = \frac{2(4n+1)!}{(n+1)!(3n+2)!}.$$

Sources: see Gilles Schaeffer's PhD thesis, pages 35-37, and references therein.

Exercise 4 – The slice construction (almost) unleashed

In the lectures we have seen the slice construction for quadrangulations. In this exercise we will consider the case of arbitrary bipartite maps. We define as before a "slice" as a map having in the external face an oriented "base edge" (l, r) and a marked corner o, such that: 1/ the left (resp. right) boundary $\ell \to o$ (resp. $r \to o$) is a geodesic (resp. unique geodesic) and 2/ o is the only point common to these two boundaries.

The only difference with the lecture is that now inner faces are not necessarily quadrangles, but can have arbitrary even degree.

1. Show that a *planar* map is bipartite if and only if all its faces have even degree.

2. As in the lecture, we can label vertices by their distance to o. Recall why the labels of (l, r) can be only of the form $(i+1, i)$ or $(i, i+1)$.

3. As in the lecture, slices of type $(i, i+1)$ are reduced to a trivial edge with $l = o$. Consider a slice of type $(i+1, i)$, and look at the inner face to the left of the base. If it has degree $2k$, what sequence of labels can appear around this face?

4. Consider the generating function $R(z)$ of elementary slices of the non-trivial type (with base of the form $(i+1, i)$), with a weight zp_k per inner face of degree $2k$ (the lecture corresponds to the case $p_k = \mathbf{1}_{k=2}$). Show that
$$R(z) = 1 + z \sum_{k=1}^{\infty} p_k \binom{2k-1}{k} R(z)^k.$$

5. Let $[\dots]$ denote coefficient extraction. Let $n_1, n_2, \dots$ be nonnegative integers with finite sum $m = \sum_k n_k$. Give an explicit formula for $[z^m p_1^{n_1} p_2^{n_2} \dots] R(z)$. (Note: for $p_k = \mathbf{1}_{k=2}$ you should find $3^m \mathrm{Cat}(m)$ as in the lectures....) Hint: use the Lagrange inversion formula[2].

6. Deduce that the number of planar rooted bipartite maps with precisely n_k faces of degree $2k$ for each $k \geq 1$ is given by
$$\frac{2}{m(N+2-m)} \binom{m}{n_1, n_2, \dots} \prod_k \binom{2k-1}{k}^{n_k} \binom{\sum k n_k}{m-1}$$
with $m = \sum_k n_k$ and $N = \sum_k k n_k$ being respectively the number of faces and the number of edges of such maps.

7. In the case of quadrangulations, show by any means you like (Lagrange inversion, differentiation -recommended-, slices -beautiful but hard, this is the next question) that the generating function of rooted quadrangulations (without pointed vertex) is $R - zR^3$.

8. Generalize this to the general bipartite case. Show that the generating function of rooted maps (without pointed vertex) and the weights p_k, z as above is equal to $R - z \sum_{k=1}^{\infty} p_k \binom{2k-1}{k+1} R^{k+1}$.

9. Do the previous calculation directly with slices (and no calculation!)

10. Bonus: is the conclusion of question 1 true in higher genus?

11. Bonus (may require some time). Generalize the construction to non-bipartite maps. Write some (system?) of algebraic equation(s) counting maps with a full control on all face degrees (say, faces of degree k receive weigth zt_k).

[2] If $F(z) = z\phi(F(z))$ then for $m > 0$ we have $[z^m]F(z) = \frac{1}{m}[y^{m-1}]\phi(y)^m$. See e.g. the book of Flajolet and Sedgewick