

Introduction to planar maps – exercise session

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Solutions in red

Exercise 0 – Crash course on Catalan numbers (optional)

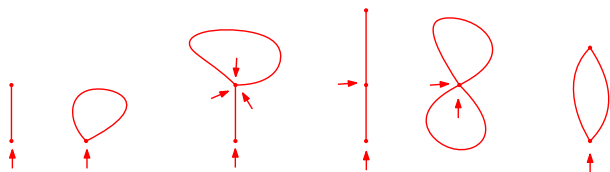
Let a_n be the number of rooted plane trees with n edges (=rooted planar maps without cycles, =rooted planar maps with a unique face) and let $A(z) = \sum_{n \geq 0} a_n z^n$.

1. Convince yourself of the following: 1/ removing the root edge in a nontrivial rooted plane tree, I naturally obtain two rooted plane trees, and I know how to go back. 2/ this implies that $A(z) = 1 + zA(z)^2$. If this is not clear to you, no problem, read an introductory course on generating functions and combinatorial classes (e.g.: Flajolet-Sedgewick).
2. Deduce that $a_n = \text{Cat}(n) := \frac{1}{n+1} \binom{2n}{n}$. (Best way to do it: apply the Lagrange inversion formula to $B(z) := A(z) - 1 = z(1 + B(z))^2$, see the footnote a bit further down).
3. Give a least one bijective proof of the Catalan number formula¹.
4. (Experts) How many (truly) different proofs of the Catalan number formula do you know?

Exercise 1 – Let's draw

1. Draw the 2 (resp. 9) rooted planar maps with 1 (resp. 2) edges, without looking at the course notes! If you feel like it, also do the 54 maps with 3 edges (I recommend you draw each unrooted map only once, and just indicate its inequivalent rootings).

Here they are, each presented with all its inequivalent rootings:



Exercise 2 – Mullin's nice formula for tree-rooted maps.

A tree-rooted map is a rooted map equipped with a distinguished spanning tree. We recall that the number of rooted plane trees (rooted planar maps with one-face, or equivalently rooted planar maps whose underlying graph is a tree) with m edges is the Catalan number $\text{Cat}(m) = \frac{1}{m+1} \binom{2m}{m}$.

1. Let M be a planar map with a spanning tree T . Show that the set of all dual-edges of edges not in T form a spanning tree of the dual map M^* .

¹most classical one: consider Dyck paths instead of rooted plane trees. A Dyck path is the list of the height of corners when you do the tour of the tree, it is a path with ± 1 steps, of length $2n$ going from 0 to 0 and staying nonnegative. A Dyck path with an extra -1 step is called a Lukaciewicz word. Now prove the cycle lemma: given any path with $(n+1)$ steps -1 and n steps $+1$ (but possibly being negative at some places), exactly one of its $(2n+1)$ cyclic conjugates is a Lukaciewicz word. This proves that $a_n = \frac{1}{2n+1} \binom{2n+1}{n}$.

Generally for a planar map M and $S \subset E(M)$, the dual submap $S^* := \{e^*, e \in S\}$ is connected (on the vertex set $V(M^*)$) if and only if S is acyclic: this is the Jordan lemma!! Since a tree is precisely a graph which is connected and acyclic, S is a tree if and only if S^* is a tree.

2. Deduce the Euler formula: $v + f = e + 2$ (we will not use it, but it's a nice consequence!).
Write the relation vertices=edges+1 for T and T^* , and sum them.
3. Convince yourself (draw a picture!) that when you follow the tour of T , you simultaneously follow the tour of T^* . Given two rooted plane trees T and T^* with a total of n edges, what sort of extra information do you need to glue them together and form a map M ?
Superimpose the drawing of M and M^* , and follow the contour of T . If T has k edges, you see a word on F, C with $2k$ letters "F" (follow and edge of T) and $2(n - k)$ letters "C" (cross an edge not in T , which you could also understand as "follow an edge of T^* "). This is precisely the gluing data needed to re-glue the two trees.
4. Show that the number of tree-rooted planar maps with n edges is equal to

$$\sum_{k=0}^n \binom{2n}{2k} \text{Cat}(k) \text{Cat}(n - k) = \text{Cat}(n) \text{Cat}(n + 1).$$

LHS comes from previous question, RHS follows by re-arranging factorials and using the combinatorially obvious identity: $\sum_{k=0}^n \binom{n}{k} \binom{n+2}{n-k} = \binom{2n+2}{n}$.

5. Bonus: Go read Olivier Bernardi's paper (or find some nice slides) for a (highly non trivial) sum-free bijective proof of the last result!
6. Bonus 2: what is the dual of a spanning tree for a map drawn on a surface of higher genus? say, a torus??? Note: For maps on higher genus surfaces we require that the faces are simply connected.
Because you can make the tour of T^* at the same time of the tour of T (this is only a "local" observation, unrelated to the global properties of the surface), T^* is a map with a unique face. But in higher genus, no Jordan lemma, so this doesn't imply that T^* is a tree!!! One-face maps on higher genus surfaces (that can be viewed as a polygon with side-identifications) are very cool combinatorial objects!

Exercise 3 – Have fun with the BMJ algorithm, counting all sorts of maps.

In the lectures (see the notes!) we have seen how to write the Tutte equation counting maps, and how to solve it. Given an equation of the form

$$E(A(z, u), a(z), z, u) = 0$$

with unknown series $A(z, u), a(z)$ the BMJ algorithm looks for a series $U = U(z)$ such that

$$(\partial_1 E)(A(z, U(z)), a(z), z, U(z)) = 0,$$

which by the chain rule also implies that

$$(\partial_4 E)(A(z, U(z)), a(z), z, U(z)) = 0.$$

We then perform algebraic elimination to find polynomial equations for $A(z, U(z)), U(z)$... and $a(z)$!

In this exercise we ask you to write the analogue of the Tutte equation for different families of planar maps (in each case you have to find what is the good choice of "catalytic" variable). You can then solve it using BMJ on your favourite computer, and deduce the corresponding numbers. As in the lecture we write $\Delta(A(z, u)) := \frac{A(z, u) - A(z, 1)}{u - 1}$, and we recall that $\Delta u^k = u^{k-1} + u^{k-2} + \dots + 1$ which is useful to "add a diagonal in a root face in all possible ways".

1. Rooted maps with n edges (done in class)

$$F(z, u) = 1 + zu^2 F(z, u)^2 + zu \Delta(u F(z, u)) \quad , \quad f_n = \frac{2 \cdot 3^n}{n + 2} \text{Cat}(n).$$

2. Rooted bipartite maps with n edges.

$$F(z, u) = 1 + zuF(z, u)^2 + zu\Delta F(z, u) \quad , \quad f_n = \frac{3 \cdot 2^{n-1}(2n)!}{n!(n+2)!}.$$

3. Non-separable maps with n edges (i.e. maps without a cut-vertex, i.e. maps with a 2-connected graph; in particular we forbid loops)

$$F(z, u) = 1 + zuF(z, u)(1 + \Delta F(z, u)) \quad , \quad f_n = \frac{2(3n)!}{(n+1)!(2n+2)!}.$$

4. Non-separable and *cubic* maps (all vertices of degree 3) with $n+1$ faces.

$$F(z, u) = u + zu^2F(z, u)\Delta F(z, u) \quad , \quad f_n = \frac{2^n(3n)!}{(n+1)!(2n+2)!}.$$

5. Three-connected cubic maps with $n+1$ faces.

$$F(z, u) = 1 + zuF(z, u)\Delta F(z, u) \quad , \quad f_n = \frac{2(4n+1)!}{(n+1)!(3n+2)!}.$$

Sources: see Gilles Schaeffer's PhD thesis, pages 35-37, and references therein.

Sorry, I will not write all of them... Have a look at the paper "Planar maps and description trees" by Cori and Schaeffer, which lists these examples together with the parameters you have to take into account, and the decompositions.

Also, here is an example of how I solve the Tutte equation on my computer with Maple (other computer algebra systems should have similar tools. Note that here I use the `-great-` package `gfun`).

```
> ## The Tutte equation
eq:=-F + 1 + z*u^2*F^2 + z*u*(u*F-f)/(u-1);

                                eq:=-F + 1 + z*u^2*F^2 + \frac{z*u*(F*u-f)}{u-1}

> ## The BMJ system (for simplicity I take the numerators of these equations!)
system_:= {numer(eq), numer(diff(eq,F)), numer(diff(eq,u))};

                                system_:= {z (2 F^2 u^3 - 4 F^2 u^2 + 2 F^2 u + F u^2 - 2 F u + f), 2 F u^3 z - 2 F u^2 z + u^2 z - u + 1, F^2 u^3 z - F^2 u^2 z + F u^2 z - f u z - F u + F + u - 1}

> ## side remark: the third equation is of the form u = 1+ stuff(F,f,u)
## Since F and f are uniquely determined as power series, this implies
## that U(z) is uniquely determined as a power series (and exists!)
## This justifies the validity of the method!!!
## (the BMJ theorem gives you general hypotheses under which that would happen but here we just check it by hand)
> ## Now we solve the system (eliminate (u=U(z) and F=F(z,U(z))), to just get an equation of f=f(z)
eliminate(system_, {F,u}) [2,1];

                                z (27 f^2 z^2 - 18 f z + f + 16 z - 1) (f z - 1)

> ## Check
series(RootOf(27*f^2*z^2-18*f*z+f+16*z-1,f), z=0);

                                1 + 2 z + 9 z^2 + 54 z^3 + 378 z^4 + 2916 z^5 + O(z^6)

> ## extraction of coefficient via: algeq->diffeq->rec->formula
gfun[algeqtodiffeq](27*f^2*z^2-18*f*z+f+16*z-1,f(z));
gfun[diffeqtorec](%,f(z),a(n));
rsolve(%,a(n));
simplify(%/binomial(2*n,n))*binomial(2*n,n);

                                2 + (6 z - 2) f(z) + (12 z^2 - z) \left( \frac{d}{dz} f(z) \right)
                                \{ (6 + 12 n) a(n) + (-3 - n) a(n + 1), a(0) = 1 \}
                                \frac{2 \cdot 12^n \Gamma\left(n + \frac{1}{2}\right)}{\sqrt{\pi} \Gamma(3 + n)}
                                \frac{2 \cdot 3^n \text{binomial}(2 n, n)}{(n + 1) (2 + n)}
```

Exercise 4 – The slice construction (almost) unleashed

In the lectures we have seen the slice construction for quadrangulations. In this exercise we will consider the case of arbitrary bipartite maps. We define as before a "slice" as a map having in the external face an oriented "base edge" (l, r) and a marked corner o , such that: 1/ the left (resp. right) boundary $\ell \rightarrow o$ (resp. $r \rightarrow o$) is a geodesic (resp. unique geodesic) and 2/ o is the only point common to these two boundaries.

The only difference with the lecture is that now inner faces are not necessarily quadrangles, but can have arbitrary even degree.

1. Show that a *planar* map is bipartite if and only if all its faces have even degree.
1/ If the map is bipartite, corners alternate white/black around each face, so face degrees are even. 2/ A graph is bipartite if and only all simple cycles are even. Now in a planar map, by the Jordan curve lemma, a simple cycle separates a finite region, which is a union of faces. The sum of face degrees inside this region is congruent modulo 2 to the length of the cycle – because each inner edge is counted twice. So if face degrees are even, all cycles are even and we are done.
2. As in the lecture, we can label vertices by their distance to o . Recall why the labels of (l, r) can be only of the form $(i + 1, i)$ or $(i, i + 1)$.
Since the map is bipartite, labels vary by 1 modulo 2 around an edge, and by the triangle inequality, they vary by at most 1. So they vary by *exactly* one.
3. As in the lecture, slices of type $(i, i + 1)$ are reduced to a trivial edge with $l = o$. Consider a slice of type $(i + 1, i)$, and look at the inner face to the left of the base. If it has degree $2k$, what sequence of labels can appear around this face?
You see a path of length $2k - 1$ with steps $+1$ or -1 , going from $i + 1$ to 1. Along this path there are k steps -1 and $k - 1$ steps $+1$.
4. Consider the generating function $R(z)$ of elementary slices of the non-trivial type (with base of the form $(i + 1, i)$), with a weight zp_k per inner face of degree $2k$ (the lecture corresponds to the case $p_k = \mathbf{1}_{k=2}$). Show that

$$R(z) = 1 + z \sum_{k=1}^{\infty} p_k \binom{2k-1}{k} R(z)^k.$$

This is the same as in the lecture! We remove the base edge, and around the face we throw the leftmost geodesic from each edge to o . This splits the map into $2k - 1$ slices, with $k - 1$ of them trivial and k of them "of type $R(z)$ ". We can reconstruct the original map by gluing these slices back, provided we have knowledge of the ± 1 walk from the previous question. But there are $\binom{2k-1}{k}$ such walks, so we are done!

5. Let $[\dots]$ denote coefficient extraction. Let $n_1, n_2, \dots$ be nonnegative integers with finite sum $m = \sum_k n_k$. Give an explicit formula for $[z^m p_1^{n_1} p_2^{n_2} \dots] R(z)$. (Note: for $p_k = \mathbf{1}_{k=2}$ you should find $3^m \text{Cat}(m)$ as in the lectures....) Hint: use the Lagrange inversion formula².

By the Lagrange inversion formula (applied to the series $R(z) - 1$), our coefficient is given by (we assume $m > 0$)

$$\begin{aligned} [z^m p_1^{n_1} p_2^{n_2} \dots] R(z) &= \frac{1}{m} [y^{m-1}] [p_1^{n_1} p_2^{n_2} \dots] \left(\sum_{k=1}^{\infty} p_k \binom{2k-1}{k} (1+y)^k \right)^m \\ &= \frac{1}{m} [y^{m-1}] \binom{m}{n_1, n_2, \dots} \prod_k \binom{2k-1}{k}^{n_k} (1+y)^{\sum_k kn_k} \\ &= \frac{1}{m} \binom{m}{n_1, n_2, \dots} \binom{\sum kn_k}{m-1} \prod_k \binom{2k-1}{k}^{n_k}. \end{aligned}$$

Sanity check: for $p_k = \mathbf{1}_{k=2}$, we have $m = n_2$, and $\sum kn_k = 2n$, so our formula gives

$$\frac{1}{m} \binom{m}{m} \binom{2m}{m-1} \binom{2 \cdot 2 - 1}{1}^m = 3^m \text{Cat}(m).$$

6. Deduce that the number of planar rooted bipartite maps with precisely n_k faces of degree $2k$ for each $k \geq 1$ is given by

$$\frac{2}{m(N+2-m)} \binom{m}{n_1, n_2, \dots} \prod_k \binom{2k-1}{k}^{n_k} \binom{\sum kn_k}{m-1}$$

²If $F(z) = z\phi(F(z))$ then for $m > 0$ we have $[z^m]F(z) = \frac{1}{m} [y^{m-1}] \phi(y)^m$. See e.g. the book of Flajolet and Sedgewick

with $m = \sum_k n_k$ and $N = \sum_k kn_k$ being respectively the number of faces and the number of edges of such maps.

As in the lecture, if we glue the left and right boundary in an elementary slice counted by $R(z)$, we obtain a rooted bipartite map with a pointed vertex: the converse construction consists in "cutting the map open" along the leftmost geodesic starting from the root edge to that vertex. More precisely, as in the lecture, we obtain only half of these maps (the ones with root edge oriented towards the pointed vertex). Moreover, by Euler's formula, the number v of vertices in a map with n_k faces of degree $2k$ for all $k \geq 1$ satisfies:

$$v + \sum_k n_k = \sum_k kn_k + 2.$$

So $v = N + 2 - m$ and this is the number of ways to choose a "pointed vertex", in a map with such face-type.

Putting everything together, the number we look for is

$$\frac{2}{N + 2 - m} \times (\text{result from the previous question}),$$

(factor of 2 because the slices give only half the maps, denominator because we need to "forget" the pointed vertex o). We are done!

7. In the case of quadrangulations, show by any means you like (Lagrange inversion, differentiation -recommended-, slices -beautiful but hard, this is the next question) that the generating function of rooted quadrangulations (without pointed vertex) is $R - zR^3$.

We just have to check that the coefficient of $[z^n]$ in $R - zR^3$ is $2/(n+2)Cat(n)$, we could do that by Lagrange inversion.

But here is another way that also enables one to "discover" the formula. Recall that a quadrangulation with n edges has $n + 2$ vertices. Let us look at the series

$$\tilde{R} := v(R(vz)) = v + 3z\tilde{R}^2,$$

which counts quadrangulations with a weight of $z^n v^{n+1}$ instead of z^n . In order to "unmark" the marked vertex, we have to multiply by $\frac{1}{n+2}$, which at the level of generating functions is covered by the operator $\int_0^1 dv$. Now from $\tilde{R} = v + 3z\tilde{R}^2$ we get (with z fixed) $dv = d\tilde{R} - 6z\tilde{R}d\tilde{R}$, and finally

$$\int_0^1 2\tilde{R}(z)dz = \int_0^1 2\tilde{R}(1 - z6\tilde{R})d\tilde{R} = R^2 - 4zR^3 = R(R - 4zR^2) = R(1 - zR^2).$$

(in the very last step we have replaced the first occurrence of R inside the parenthesis by the RHS of its defining equation).

8. Generalize this to the general bipartite case. Show that the generating function of rooted maps (without pointed vertex) and the weights p_k , z as above is equal to $R - z \sum_{k=1}^{\infty} p_k \binom{2k-1}{k+1} R^{k+1}$.

We similarly let

$$\tilde{R} = v + z \sum_{k=1}^{\infty} p_k \binom{2k-1}{k} \tilde{R}^k.$$

and we check that $\tilde{R}$ counts maps with the same weight as before but with an extra weight

v^{N+1-m} . As before to "unpoint" the map we just have to compute $\int_0^1 2\tilde{R}dv$. We get:

$$\begin{aligned}
\int_0^1 2\tilde{R}dv &= \int_0^1 2\tilde{R}(1 - z \sum_{k=1}^{\infty} k p_k \binom{2k-1}{k} \tilde{R}^{k-1}) d\tilde{R} \\
&= R(R - z \sum_{k=1}^{\infty} \frac{2k}{k+1} p_k \binom{2k-1}{k} R^k) \\
&= R(1 + z \sum_{k=1}^{\infty} p_k \left(1 - \frac{2k}{k+1}\right) \binom{2k-1}{k} R^k) \\
&= R(1 - z \sum_{k=1}^{\infty} p_k \frac{k-1}{k+1} \binom{2k-1}{k} R^k) \\
&= R(1 - z \sum_{k=1}^{\infty} p_k \binom{2k-1}{k+1} R^k)
\end{aligned}$$

9. Do the previous calculation directly with slices (and no calculation!)

There is an incredible trick, see the original slice paper: Bouttier Guitter, Planar maps and continued fractions.

10. Bonus: is the conclusion of question 1 true in higher genus?

No! Take $n \times m$ grid and glue opposite sides. This is a nice quadrangulation on the torus, but it is bipartite only if both m and n are even! (cultural note: a random quadrangulation of genus g is bipartite with asymptotic probability $2^{-2g} + o(1)$, so this example has some truth to it).

11. Bonus (may require some time). Generalize the construction to non-bipartite maps. Write some (system?) of algebraic equation(s) counting maps with a full control on all face degrees (say, faces of degree k receive weight zt_k).

The slice construction works exactly the same! The only difference is that, if maps are not necessarily bipartite, nothing prevents the labels along an edge to be of type (i, i) . Therefore we now have to consider three types of slices: 1/ slices with base edge of type $(i+1, i)$; 2/ slices with base edge $(i, i+1)$, as before, this is only the trivial "single edge" slice; 3/ slices with base edge of type (i, i) . We let $R(z)$ and $S(z)$ be the generating function for the first and last types. We can now apply the same decomposition as before, except that now we have to consider arbitrary paths with steps $+1, 0, -1$ (corresponding to possible variation of labels around each explored face), rather than just ± 1 . We *directly* obtain:

$$\begin{cases} R(z) = 1 + z \sum_k t_k \sum_{w \in W_k^{1 \rightarrow 0}} R(z)^{n_-(w)} S(z)^{n_0(w)} \\ S(z) = z \sum_k t_k \sum_{w \in W_k^{0 \rightarrow 0}} R(z)^{n_-(w)} S(z)^{n_0(w)} \end{cases} \quad (1)$$

where $W_k^{i \rightarrow j}$ is the set of walks with steps $+1, 0, -1$ going from i to j , and where $n_-(w)$ and $n_0(w)$ denote respectively the number of -1 steps and 0 steps of a walk w .

Now, as before, we will glue a slice back to form a map, but this requires a bit of care. Starting from a rooted and pointed map whose root edge has distances to the pointed vertex of the form $(i, i+1)$ or $(i+1, i)$, we can cut it open along the leftmost geodesic and obtain a slice. Such cases are counted by $2R(z)$ as before. For the case (i, i) , we have to split along *two* geodesics, on each side of the edge. We thus obtain two slices "of type $S(z)$ ", to be glue back together to re-form the map.

Conclusion: The generating function of rooted maps with a pointed vertex with weight zt_k for face of size k is given by

$$2R(z) + S(z)^2,$$

with R, S given by the system above. This construction goes back again to the first "slice" paper, Planar maps and continued fractions, by Bouttier and Guitter, see also this paper for references (these formulas were known before by other means).

This may look complicated... but this is super explicit. For example, the case of triangulations $t_k = \mathbf{1}_{k=3}$ gives a super simple set of equations. With a few simple calculations you obtain the following formula (originally due to Tutte) for the number $\mathbf{tri}_n$ of rooted planar triangulations with $2n$ faces

$$\mathbf{tri}_n = \frac{2^{2n+1}(3n)!!}{(n+2)!n!!}.$$

Note: the "labelled-tree-construction-à-la-Schaeffer" version of slices when you control all face degrees is called the BDG bijection after Bouttier, Di Francesco, Guitter (and the trees are called "mobiles"). The slice construction (Bouttier Guitter) is more recent.