PLANAR MAPS
(Randomness & Enumeration)
↑ just a bit ↑ mostly

guillaume.chapuy@irif.fr

Introduction to very active topic.

Will not talk about:
- higher genus
- Mathematical physics
- Matrix integrals
- Rep° theory / alg. combinatorics
- ...

Mostly: counting maps by 1/ Functional equations
2/ Surgery (= bijections)

- basic consequences in probability

Refs: see course page on my website

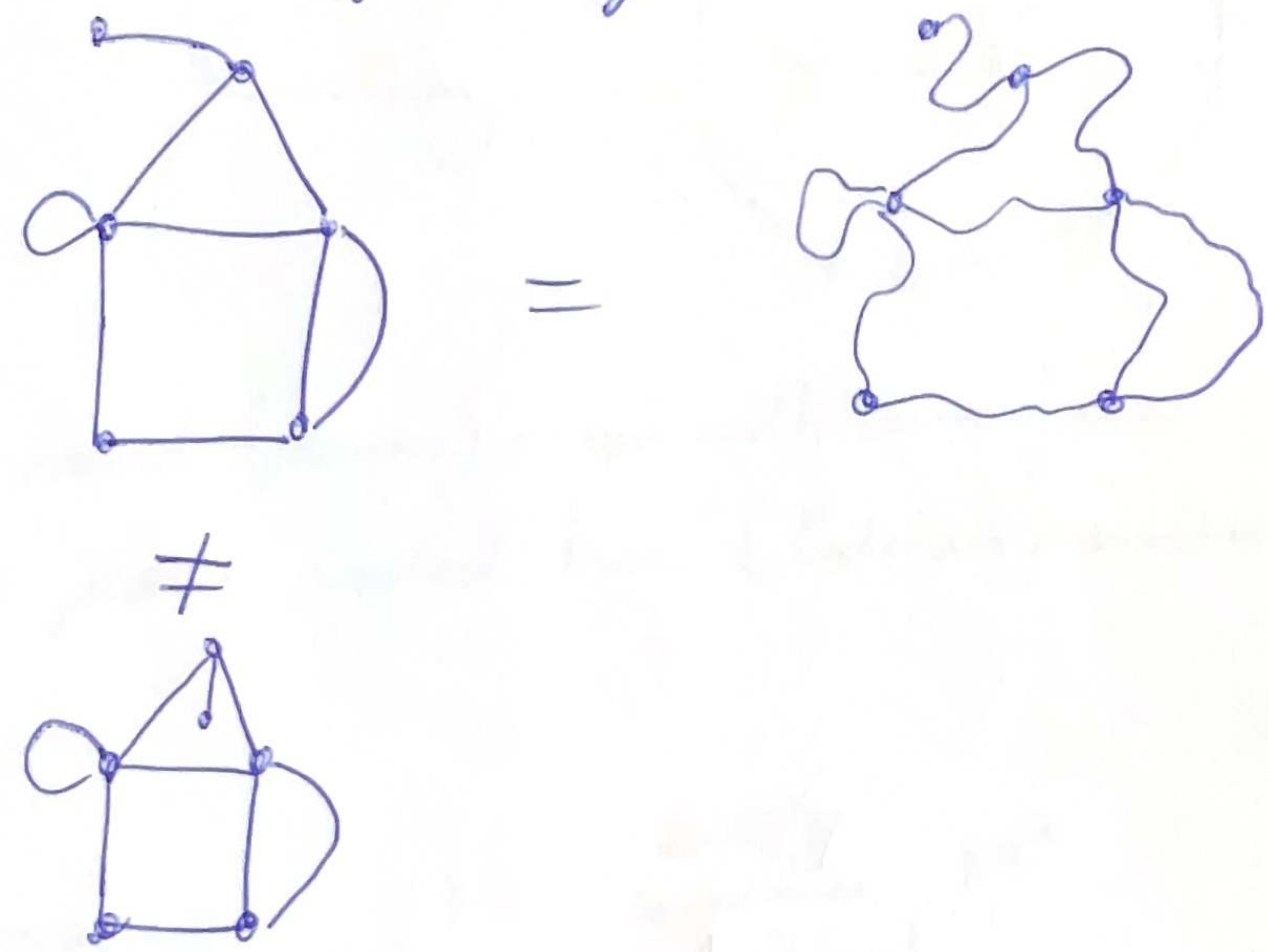
SFB Combinatorics
& Probability
Summer School

Pörtschach am Wörthersee
Austria, Sept 2025

I / Planar maps

planar map = embedding of a finite graph (loops & multiple edges allowed) in the oriented sphere, up to homeomorphism.

ex:

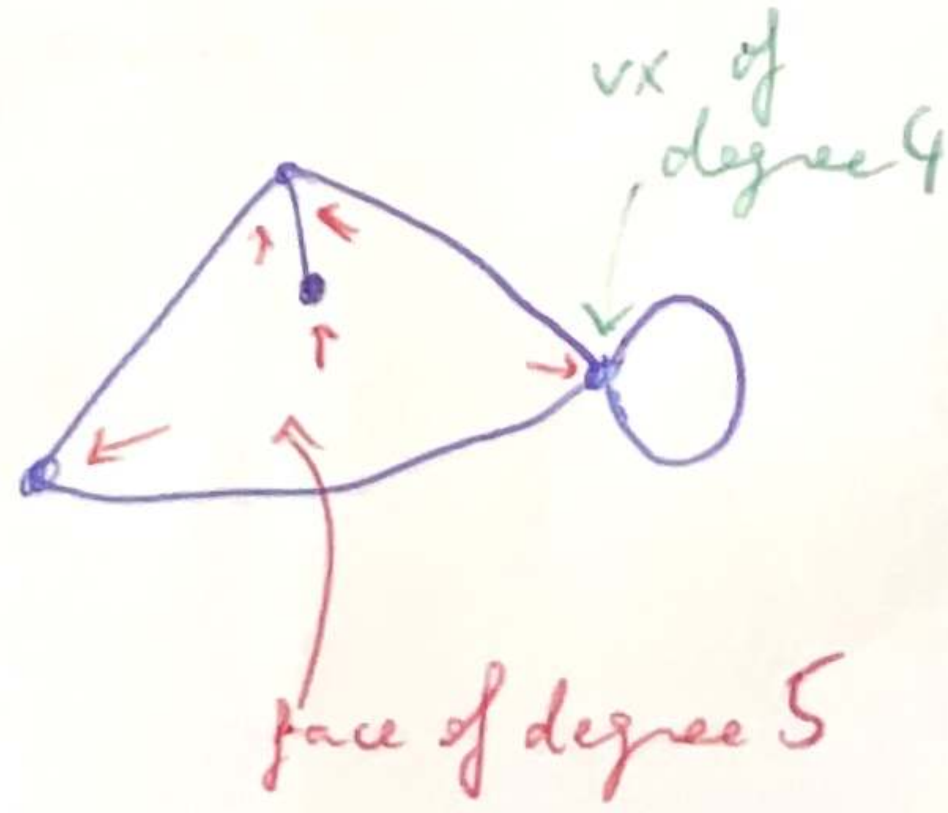


Fact: a map is determined by the data of the graph + ccw ordering of edges around each vertex. (but not all orderings give valid planar maps!)

A map has:
- vertices
- edges
- faces (conn. comp. of sphere \ graph)
- half edges ($\simeq$ incidence vertex-edge)
- corners (angular sector between two half-edges around a vertex)

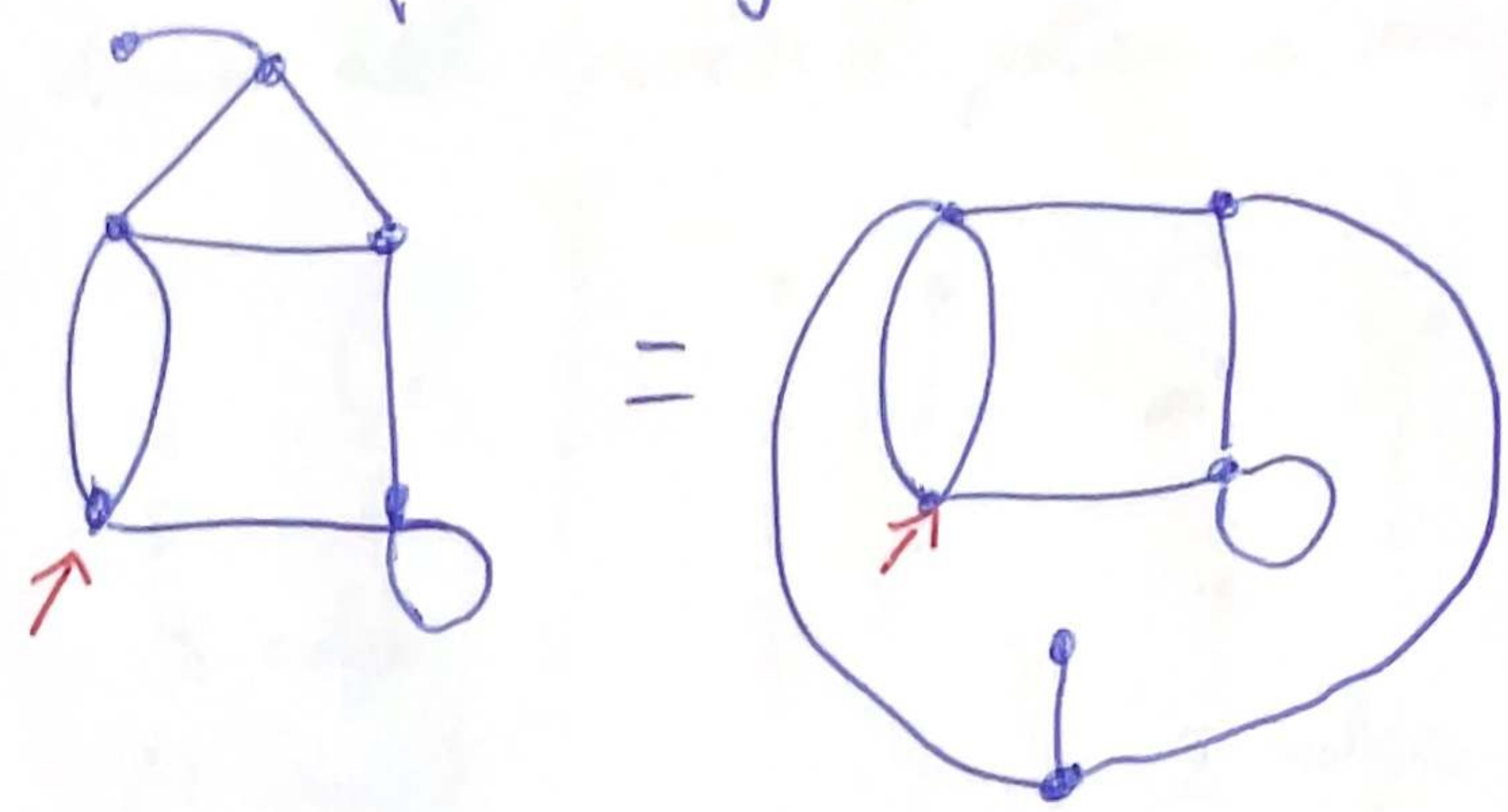
Degree – of a vertex = # incident half-edges
" corners

– of a face = # corners

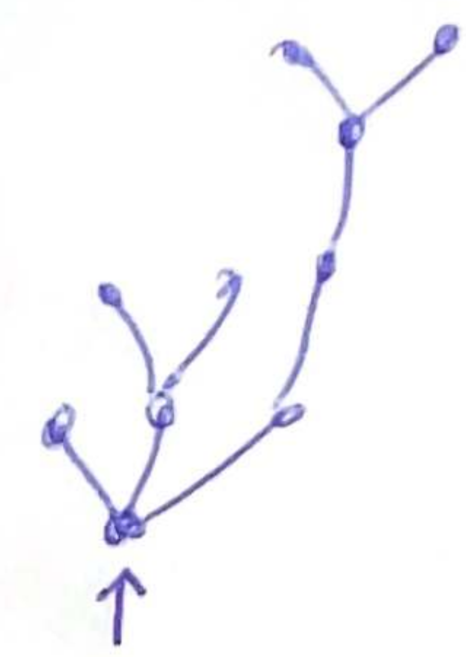


A map with n edges has $2n$ corners

Rooted map = a distinguished corner
(usually drawn on the "exterior" face)

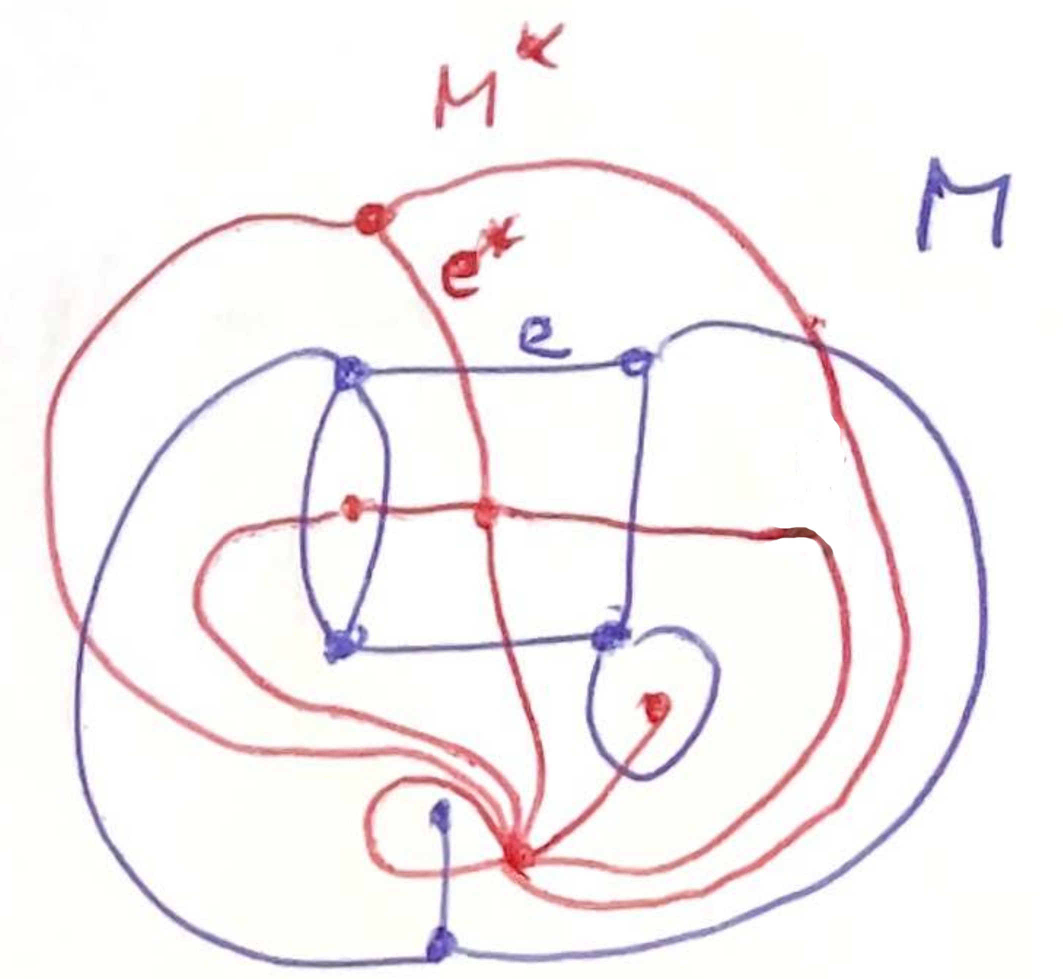


Note: rooted (planar) map with one face
= plane rooted tree (Catalan numbers!)



Dual map: M $\xleftrightarrow[\text{(involution)}]{\text{duality}}$ M^*

M	M^*
vertices	faces
faces	vertices
edges	edges



Dual submaps: M maps, $S \subseteq E(M)$
$S^* = \{e^*, e \in S\} \subseteq E(M^*)$

Fact: S^* induces a connected graph on $V(M^*)$ iff S has no cycle
(Jordan lemma)

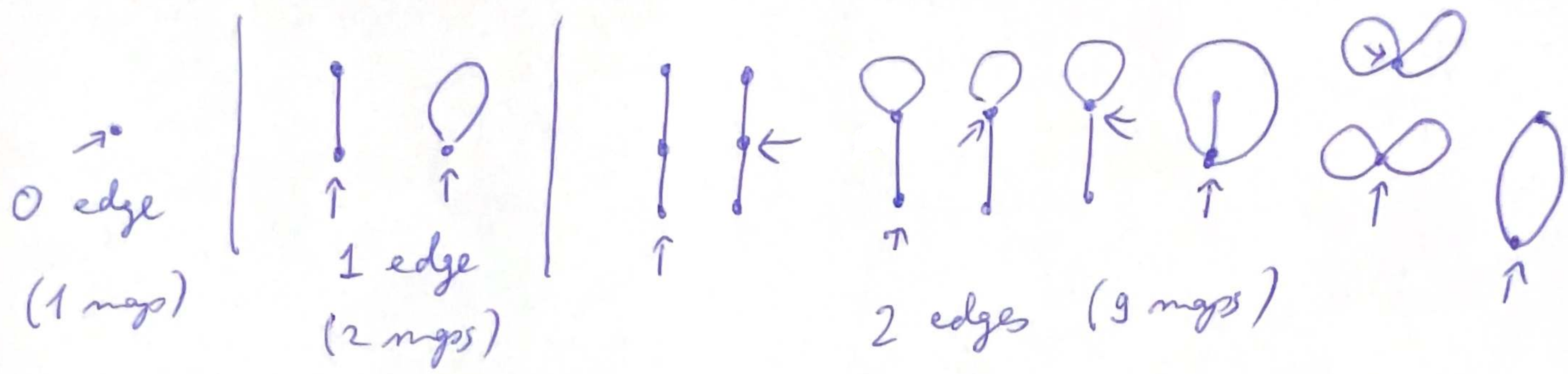
Cor1: S spanning tree of M $\Longleftrightarrow$ S^* spaning tree of M^*

Cor2: Euler's formula $V + F = E + 2$

↳ **pf:** let S be a spanning tree of M.
$1 + E(S) = V(M) = V$
$1 + E(S^*) = V(M^*) = F$
$\Rightarrow 2 + E = V + F$ $\square$

II / Counting

Let's draw all rooted planar maps up to 2 edges:



Warm-up (see exercise session)

The number of tree rooted maps (rooted maps equipped with a spanning tree) is $\mathrm{Cat}(n)\,\mathrm{Cat}(n+1)$.

Let m_n = nb of rooted planar maps with n edges

$(m_n)_{n\geq 0} = 1, 2, 9, 54, \dots$

Thm (Tutte, 1960s) (the real thing)

$$m_n = \frac{2}{n+2} 3^n \,\mathrm{Cat}(n)$$

Equivalently $M(z) = \sum m_n z^n$ is given by $\begin{cases} M = R - zR^3 \\ R = 1 + 3zR^2 \end{cases}$

COR: $m_n \sim \frac{2}{\sqrt{\pi}} n^{-5/2} 12^n$

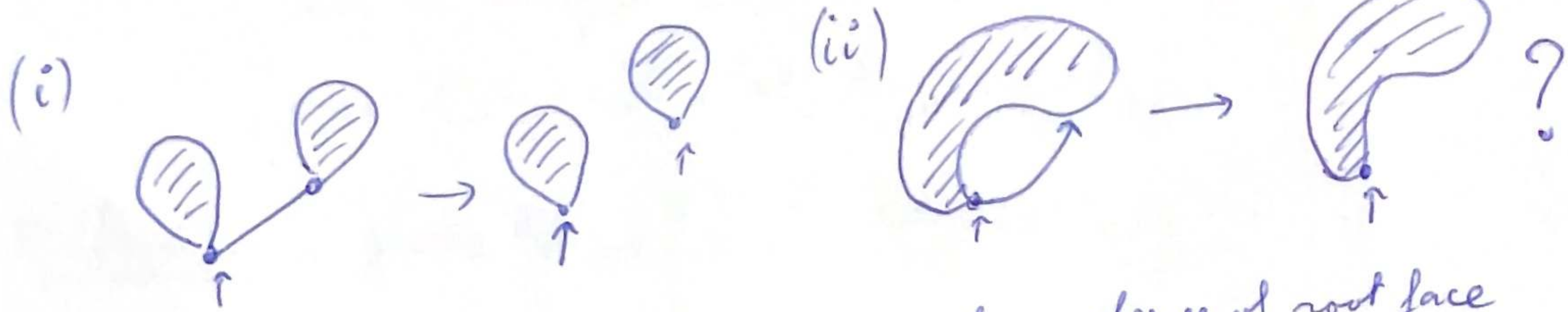
↳ "planar maps exponent"

Proof by Tutte equation . . .

Tutte equation

Let us remove the root edge in a map.

Two cases → disconnecting or not disconnecting.



Let $M(z,u) := \sum_{\substack{\text{rooted}\\ \text{planar maps}}} z^{\text{edges}}\, u^{\text{degree of root face}}$

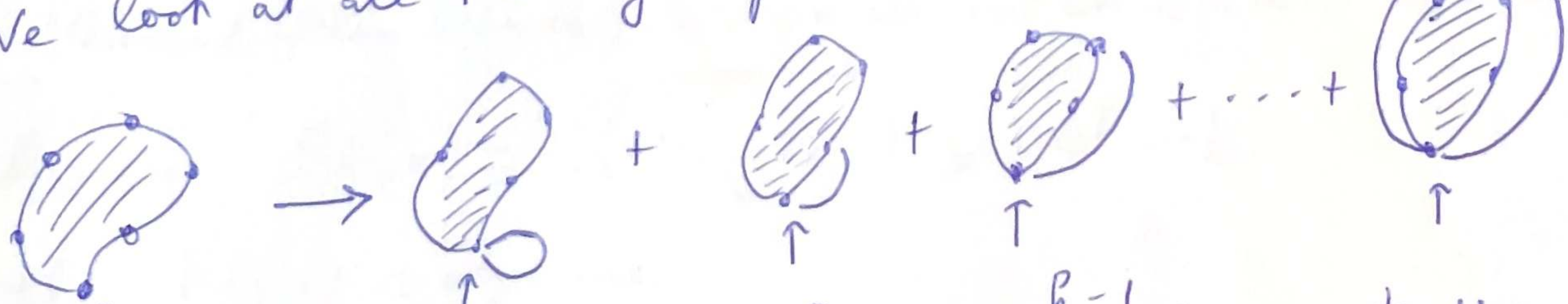
"catalytic variable"

<u>Fact</u>: we can write an equation for $M(z,u)$!

Contribution of (i): $z u^2 M(z,u)^2$

(ii): ?

We look at all the way to put back a diagonal in the root face:



$$u^k \longrightarrow u^{k+1} + u^k + u^{k-1} + \dots + u$$

$$= u\,\frac{u^{k+1}-1}{u-1}$$

Let $\Delta : A(u) \longmapsto \dfrac{A(u)-A(1)}{u-1}$. Then we have proved:

$$(*) \quad \boxed{N = 1 + z u^2 N^2 + z u\, \Delta(uN)}$$

Tutte equation!

$$N(z,u) = 1 + z u^2 N(z,u)^2 + z u\, \frac{uN(z,u) - N(z,1)}{u-1}$$

<u>Rem.</u> - $(*)$ determines a unique $N(z,u) \in \mathbb{C}[u][[z]]$

- even more: see this equation as $P(N(z,u), N(z,1), z, u) = 0$ with a polynomial P, then $P(A(z,u), a(z), z, u)$ has a unique solution in $\mathbb{C}[u][[z]] \times \mathbb{C}[[z]] \ni (A,a)$ and it is such that $A(z,1) = a(z)$

<u>Method 1</u> (Tutte)
- guess $N(z,1) = \sum \frac{2 \cdot 3^n}{n+1} \mathrm{Cat}(n) z^n$
- solve for $N(z,u)$
- check that $N(z,u) \in \mathbb{C}[u][[z]]$

<u>Thm</u> (Popescu, 1980s)

If an equation of the form $P(A_1(z,u), \dots A_k(z,u), a_1(z), \dots a_\ell(z)) = 0$ has a unique solution in $\mathbb{C}[u][[z]]^k \times \mathbb{C}[[z]]^\ell$, then each of these series is algebraic

<u>Thm</u> (Bousquet-Mélou Jehanne, slightly weaker but comes with an algorithm!)

Assume $F(z,u) = P_0(u) + z P_1(F(z,u); \Delta F(z,u); \dots (\Delta^k F)(z,u))$

polynomials

then $F(z,u)$ is algebraic

<u>BMJ algorithm</u> (case $k=1$)

Assume $E(A(z,u), a(z), z, u) = 0$

Look for $u = U(z)$ such that $(\partial_1 E)(A(z,u), a(z), z, u) = 0$

These two equations imply $(\partial_u E)(A(z,u), a(z), z, u) = 0$

$\left(\text{since } E = 0 \Rightarrow A'(u) \partial_1 E + \partial_u E = 0\right)$

$\to$ we have a system $\begin{cases} E(A,a,z,u)=0 \\ (\partial_1 E)(A,a,z,u)=0 \\ (\partial_4 E)(A,a,z,u)=0 \end{cases}$

with tree unknowns $(A=A(z,U(z)),\ a=a(z),\ u=U(z))$.

↳ demo with Maple! (but doable by hand of course)

$\rightsquigarrow$ Tutte's formula! $\square$

<u>Complement: remarkable very recent result:</u>

<u>Thm</u> [Contat-Curien 2025+. $k=1$: C/Drmota-Noy-Yu ...]

A BMJ-type equation with nonnegative coefficient, under mild hypotheses (including non-linearity!) always has an $-\frac{5}{2}$ counting exponent:

$$[z^n]F(z,1) \underset{n\to\infty}{\sim} c\, n^{-5/2} K^n$$

(great result! Follows from deep general classification of some self-similar random trees).

III Bijections

rooted maps $|\mathcal{M}_n| = \frac{2}{n+2} 3^n \mathrm{Cat}(n)$??!
n edges

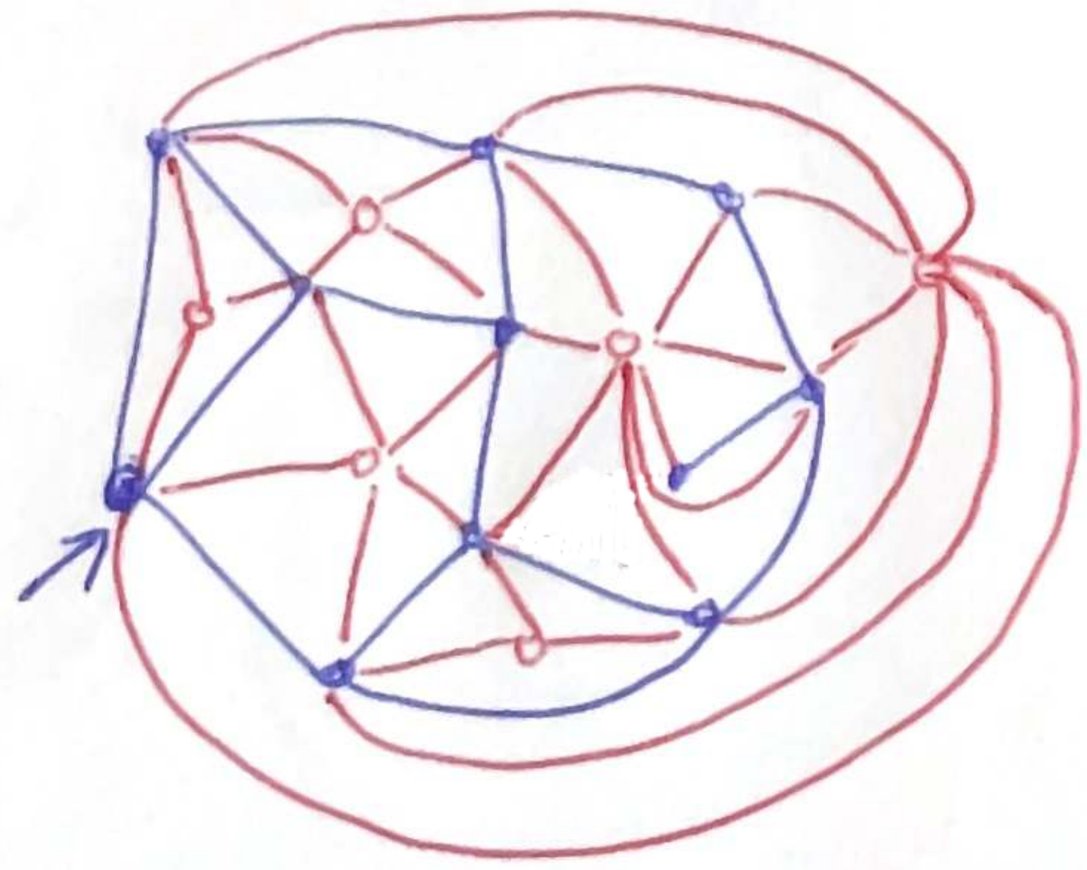
quadrangulation = map with all faces of degree 4.

$Q_n = \{$rooted quadrangulations with n faces$\}$.

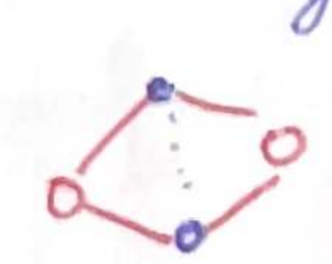
Fact: if $q \in Q_n$, q has $n+2$ vertices
(pf: $v+f=e+2$, $4f=2e$)

Fact: if $q \in Q_n$, then q is bipartite
(pf: see exercises!)

Tutte bijection: $\mathcal{M}_n \xleftrightarrow{\text{bij}} Q_n$



(forget blue edges!)



Conversely:

Let $Q_n^\bullet = \{(q,v),\ q \in Q_n,\ v \in V(q)\}$

$|Q_n^\bullet| = (n+2)|Q_n| = 2 \cdot 3^n \cdot \mathrm{Cat}(n)$??!

Let us prove this combinatorially!

Cori-Vauquelin-Schaeffer – Bouttier-Guitter ....
trees ≃ slices
1981 1998 ≥20..

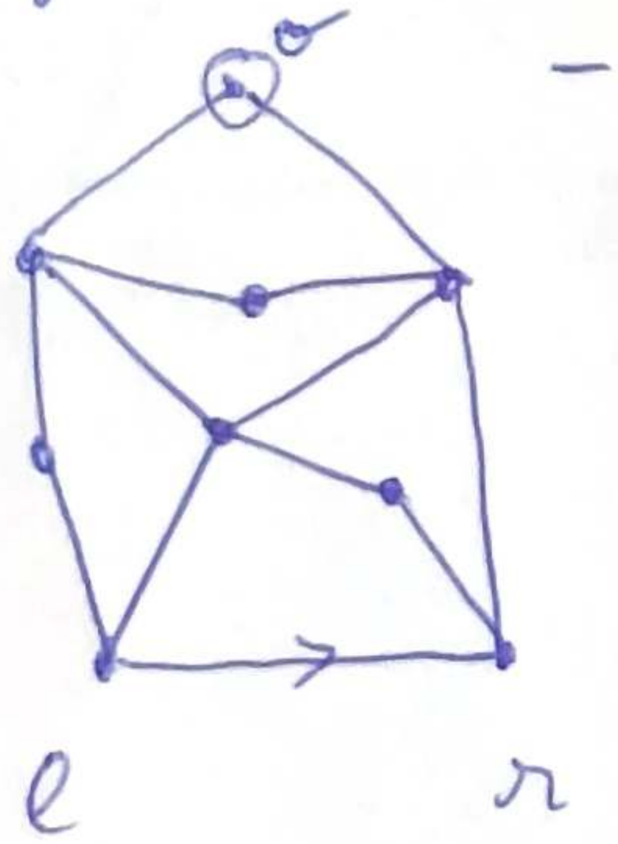
An <u>elementary slice</u> is a planar map with:

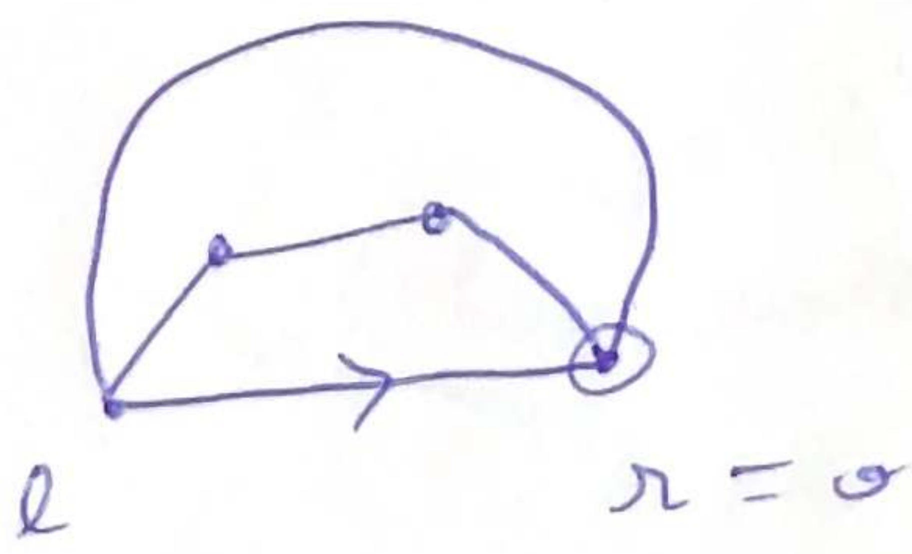
- all faces quadrangular except external one
- a "base" oriented edge $l \to r$ in the external face
- an "apex" corner σ in the external face

such that:

- left boundary is a geodesic $l \rightsquigarrow \sigma$
- right boundary is the <u>unique</u> geodesic $r \rightsquigarrow \sigma$
- apex is the only vertex belonging to both boundaries

ex:

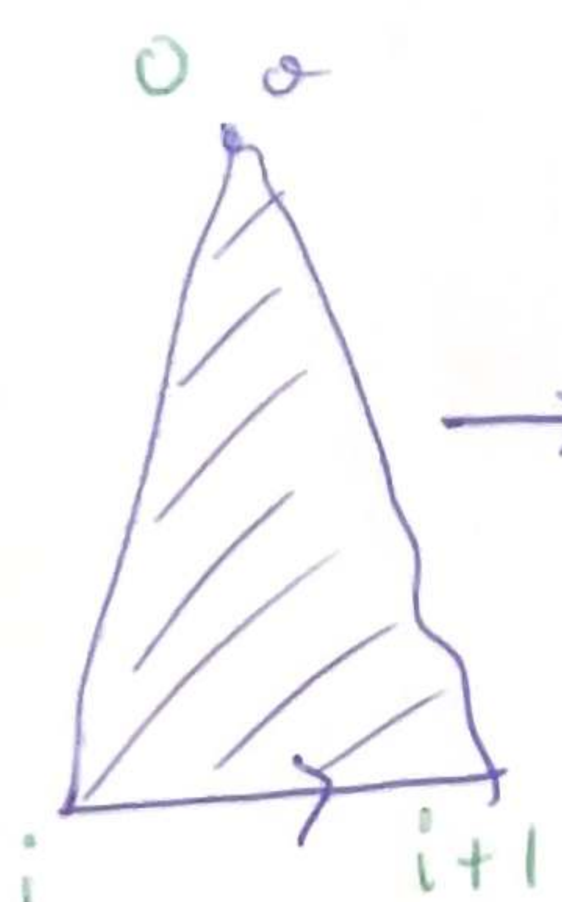




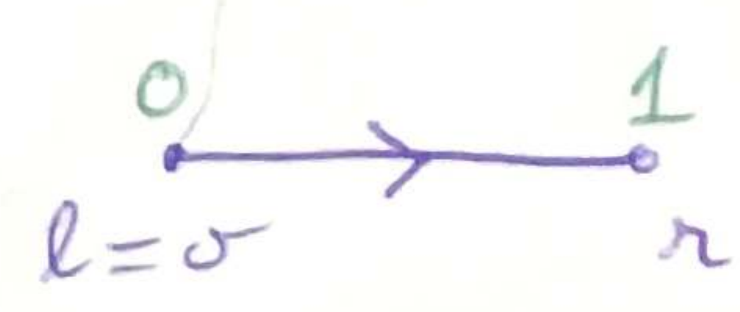
We <u>label</u> vertices by their distance to σ.

Only two cases:

or

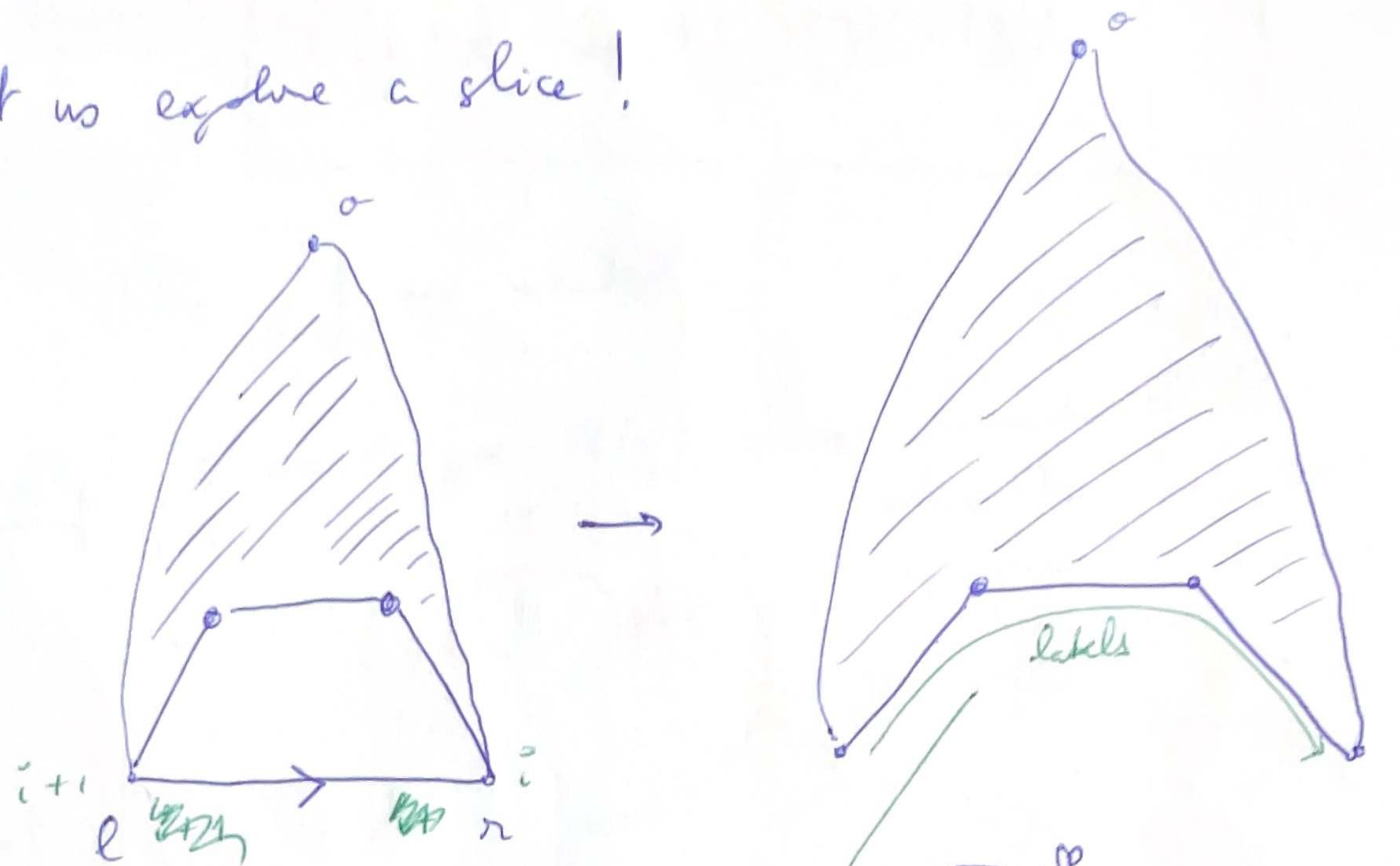


$\to$ only in the <u>one</u> degenerate case



Let $R(z)$ = generating function for the first case.

Let us explore a slice!

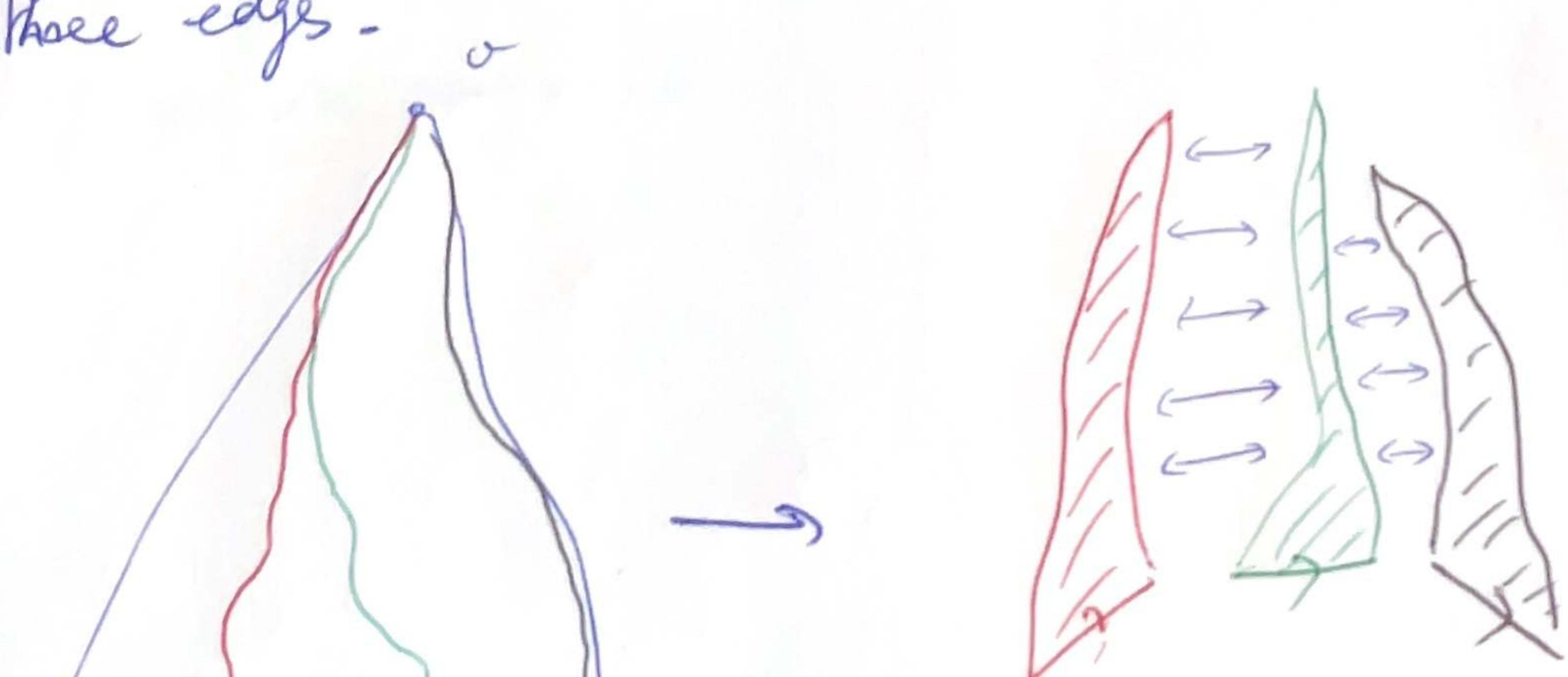


Three possible label sequences.

(*)
- $i+1 \# i+2 \# i+1 \# i$
- or $i+1 \# i \# i+1 \# i$
- or $i+1 \# i \# i-1 \# i$

↳ in each case,
- one ascent
- two descents

Further explore the <u>left-most geodesic</u> starting from each of the three edges.



Three slices!!

↳ one trivial ($i \to i+1$)
two of type "R(3)".

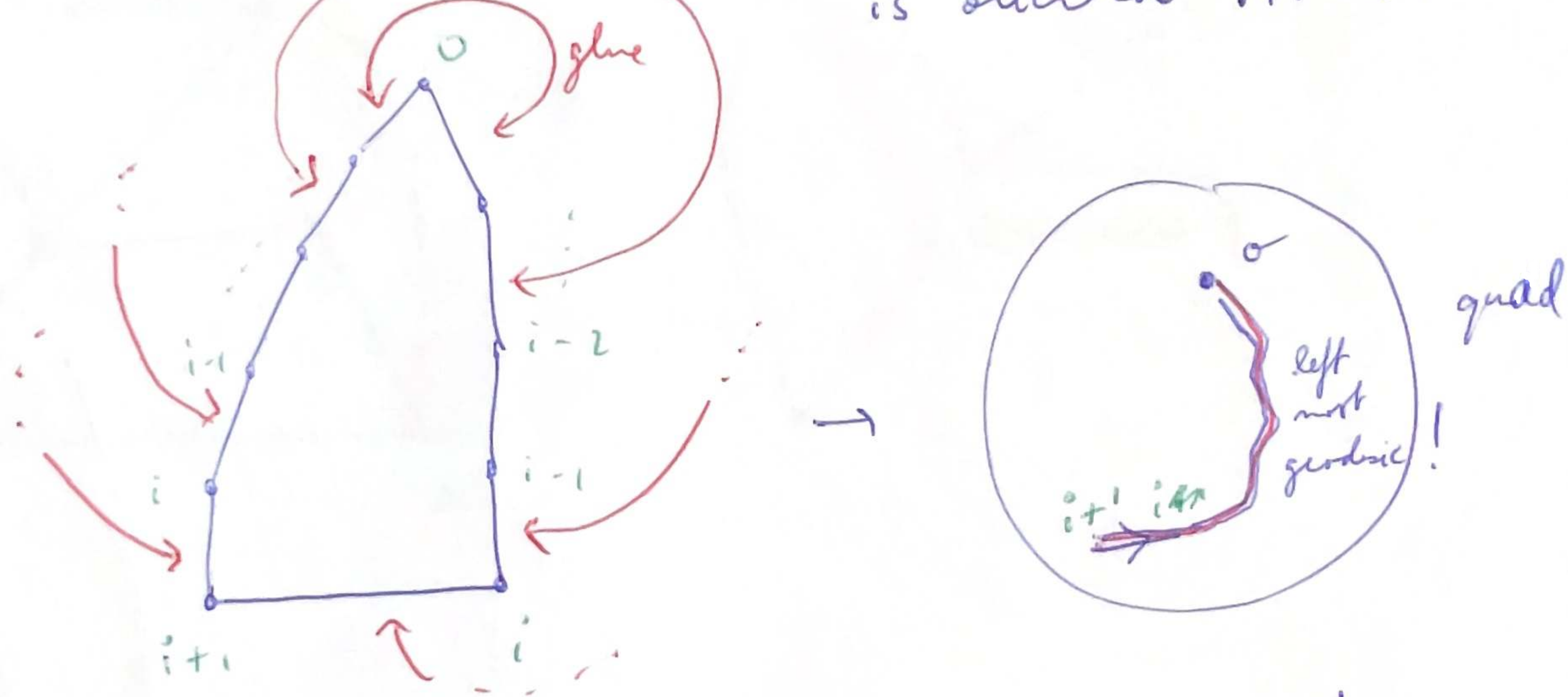
by (*)

Conclusion: $\boxed{R(z) = 1 + 3\, z\, R(z)^2}$!

the g.f we wanted!

Finally: slices of type $R(z)$ $\xleftrightarrow{\text{bij}}$ elements of $Q_n^{\bullet}$ where the root-edge is oriented $i+1 \to i$

pf:

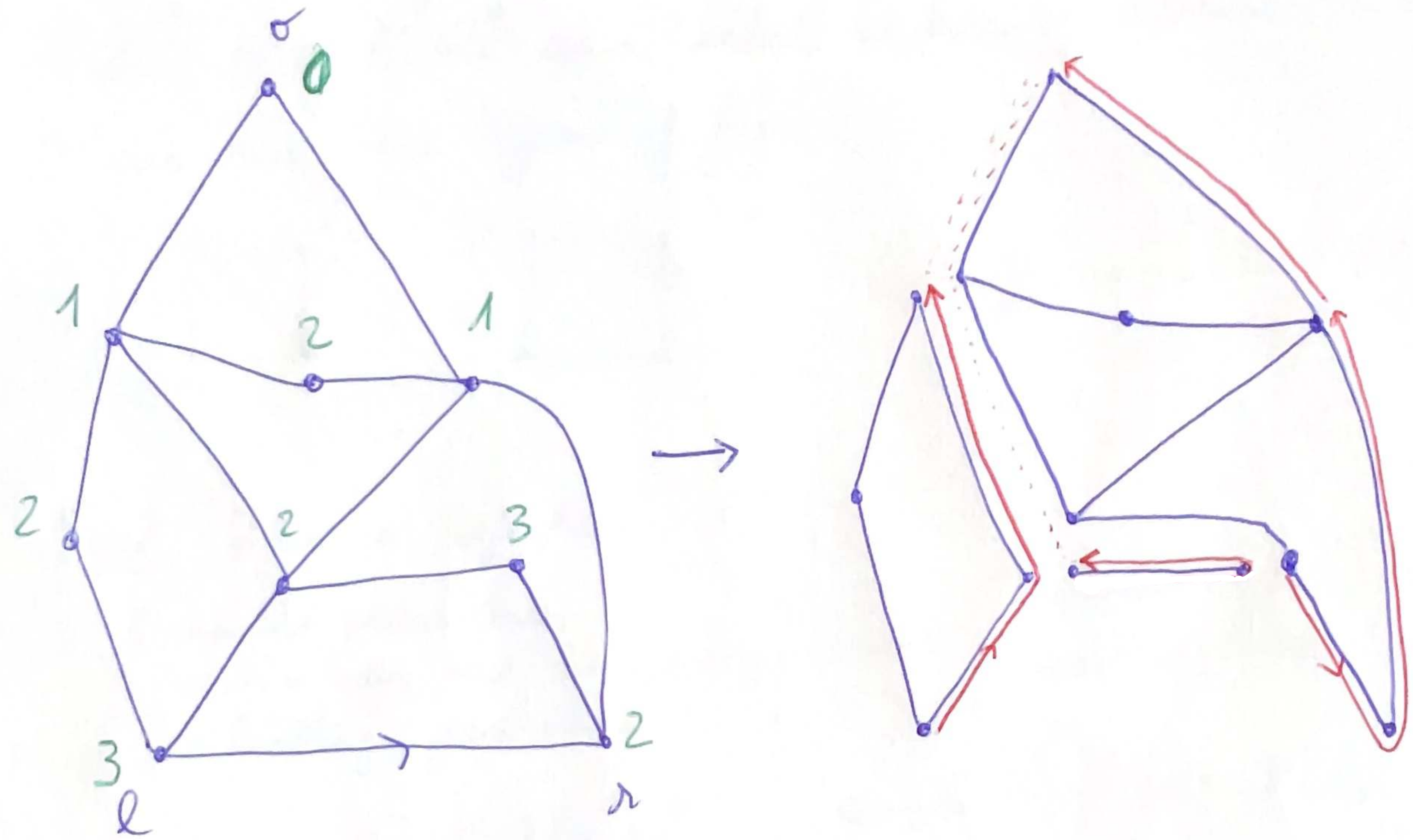


$\to \quad |Q_n^{\bullet}| = (n+2)|Q_n| = 2 \cdot 3^n \,\mathrm{Cat}(n)$!

→ generalizes almost directly to $2p$-angulations and in fact bipartite maps with all degrees controlled.

cf exercise session!!

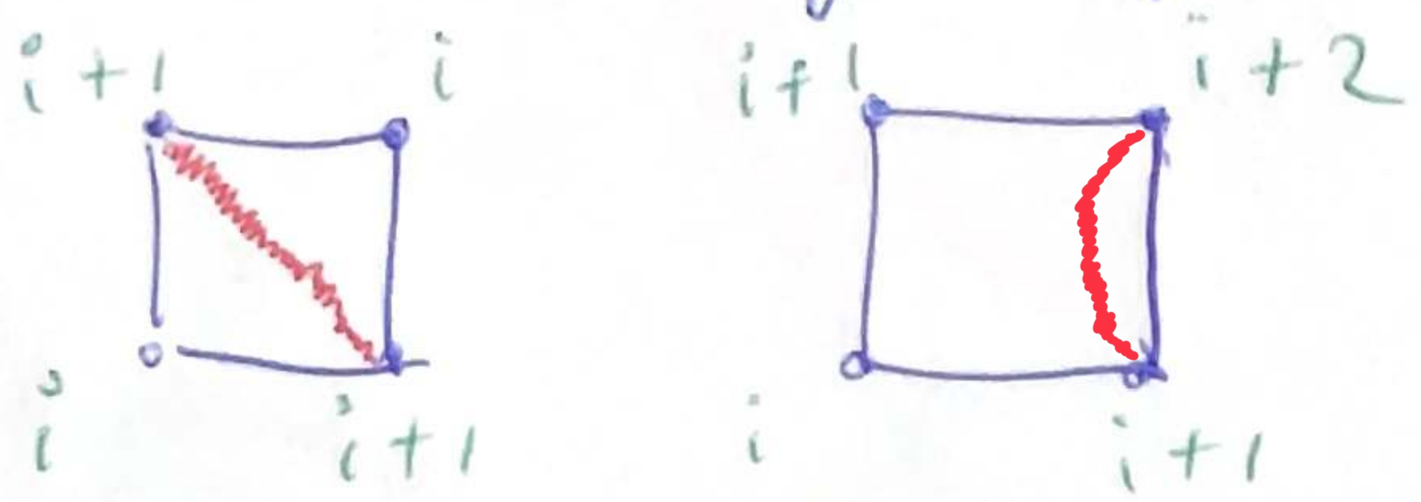
ex :



Tree interpretation (anterior-Schaeffer's bijection)

Take $(q,v) \in Q_n^*$ and label vertices by distance to v.

There are two types of faces:



Schaeffer rules: add red edge

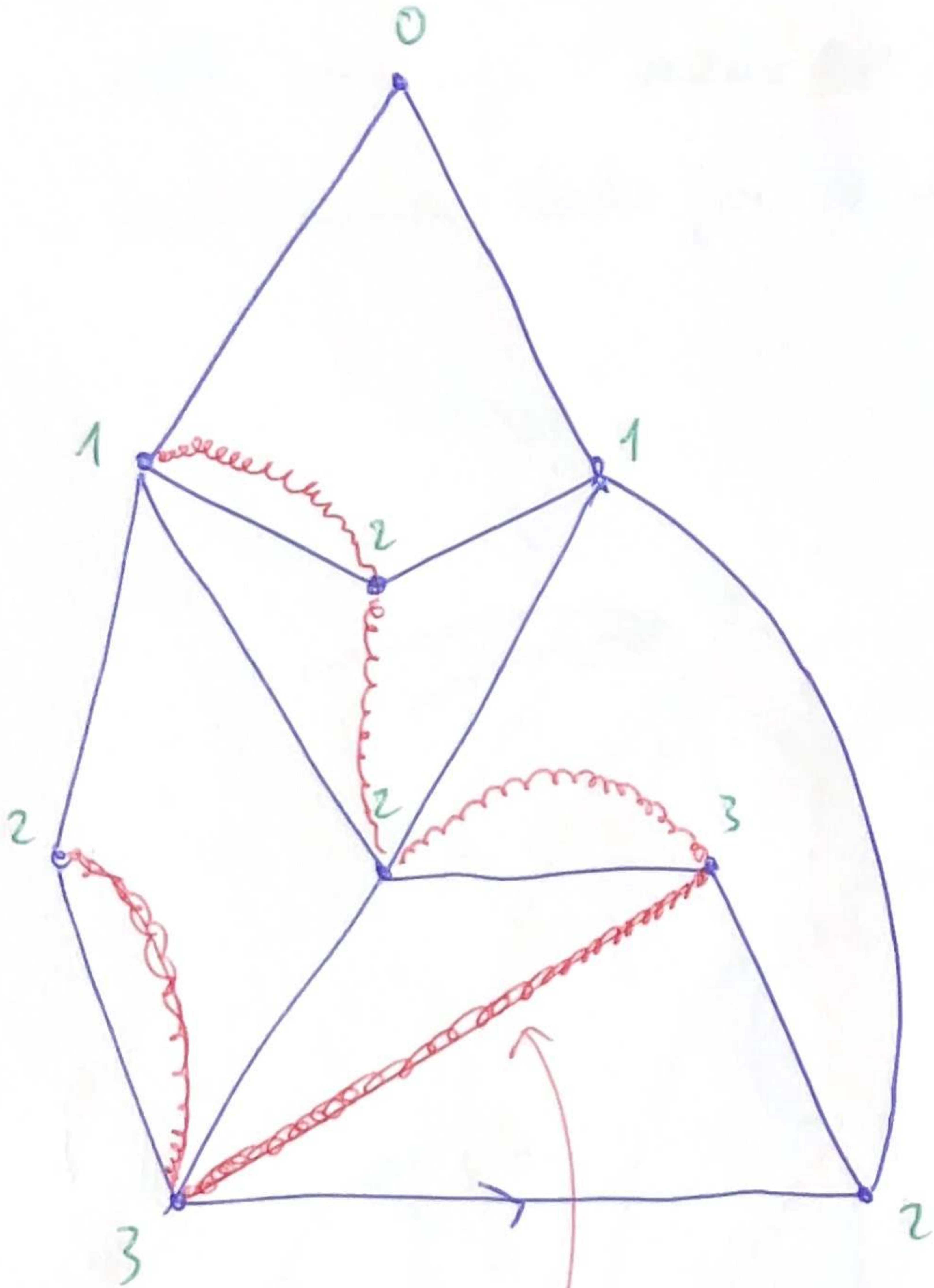
Fact: This a bijection:

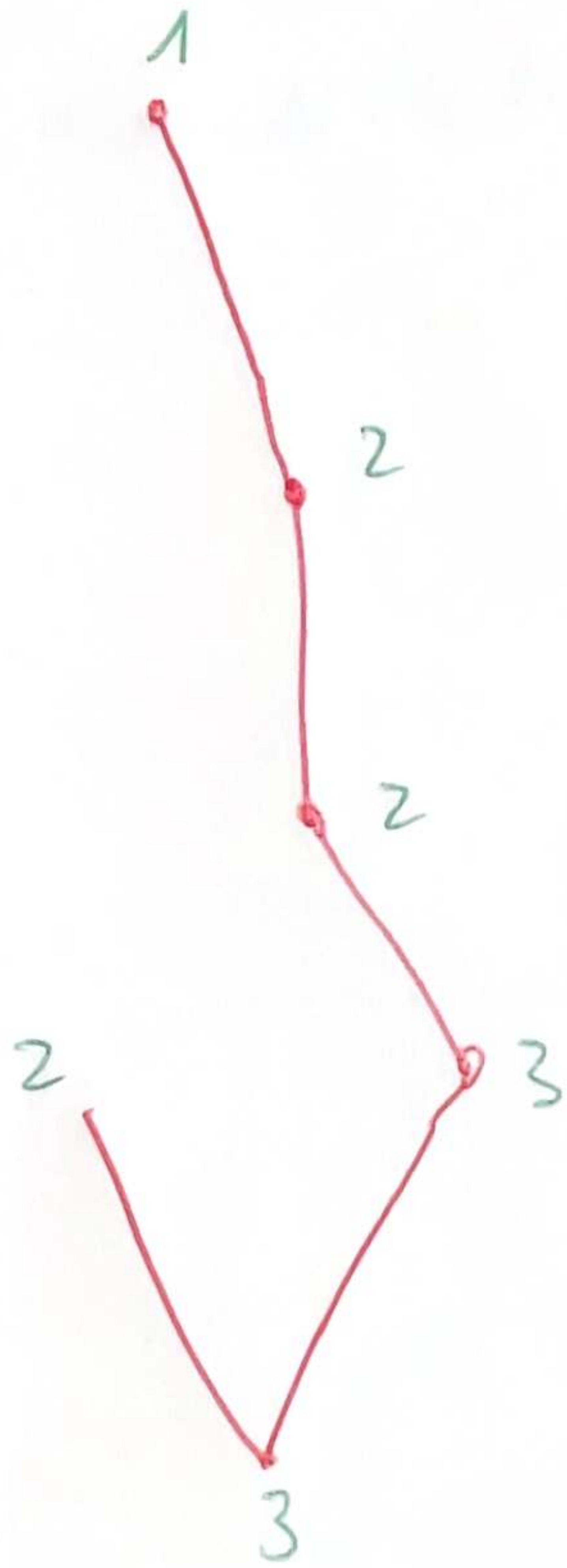
$$\left\{\begin{array}{l}\text{rooted plane trees}\\ \text{with } n \text{ edges, and 1-Lipschitz}\\ \text{vx labelling (global minim 1)}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{l}\text{elements of } Q_n^\circ \text{ with}\\ \text{root oriented } i+1 \to i\end{array}\right\}$$

$$3^n \cdot Cat(n) \qquad \longleftrightarrow \qquad \frac{1}{2}(n+2)\, Q_n$$

In fact, this is the same as the slice construction:

ex: (on a slice)

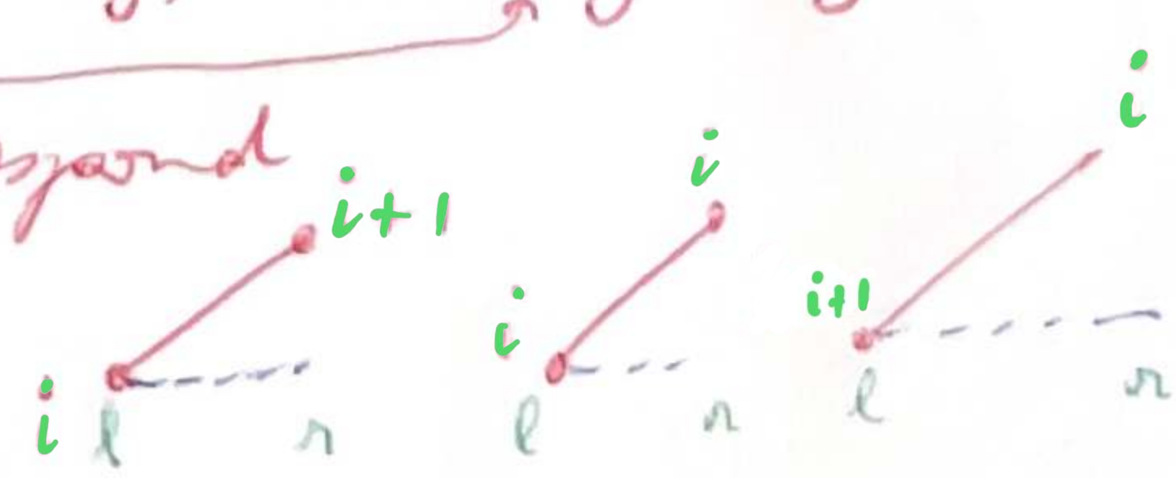




around the face, retains/links only the two corners followed by a descent (i.e. the ones that were giving rise ~~to~~ the "non-trivial" $R(z)$ slices".

→ the recursive structure of this tree = the recursive structure of the slice equation $R(z) = 1 + 3zR(z)^2$

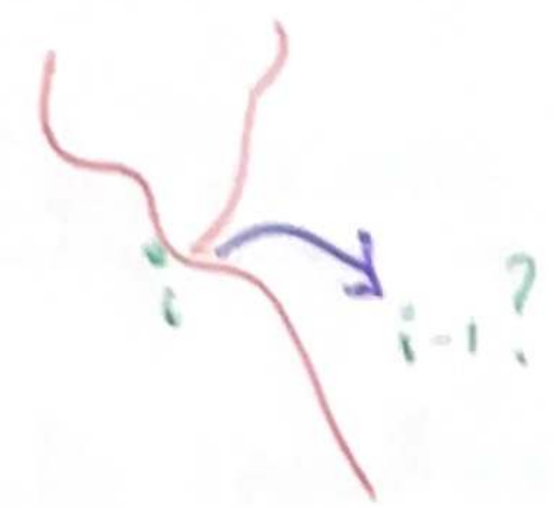
(and the three cases correspond to the three types of edges

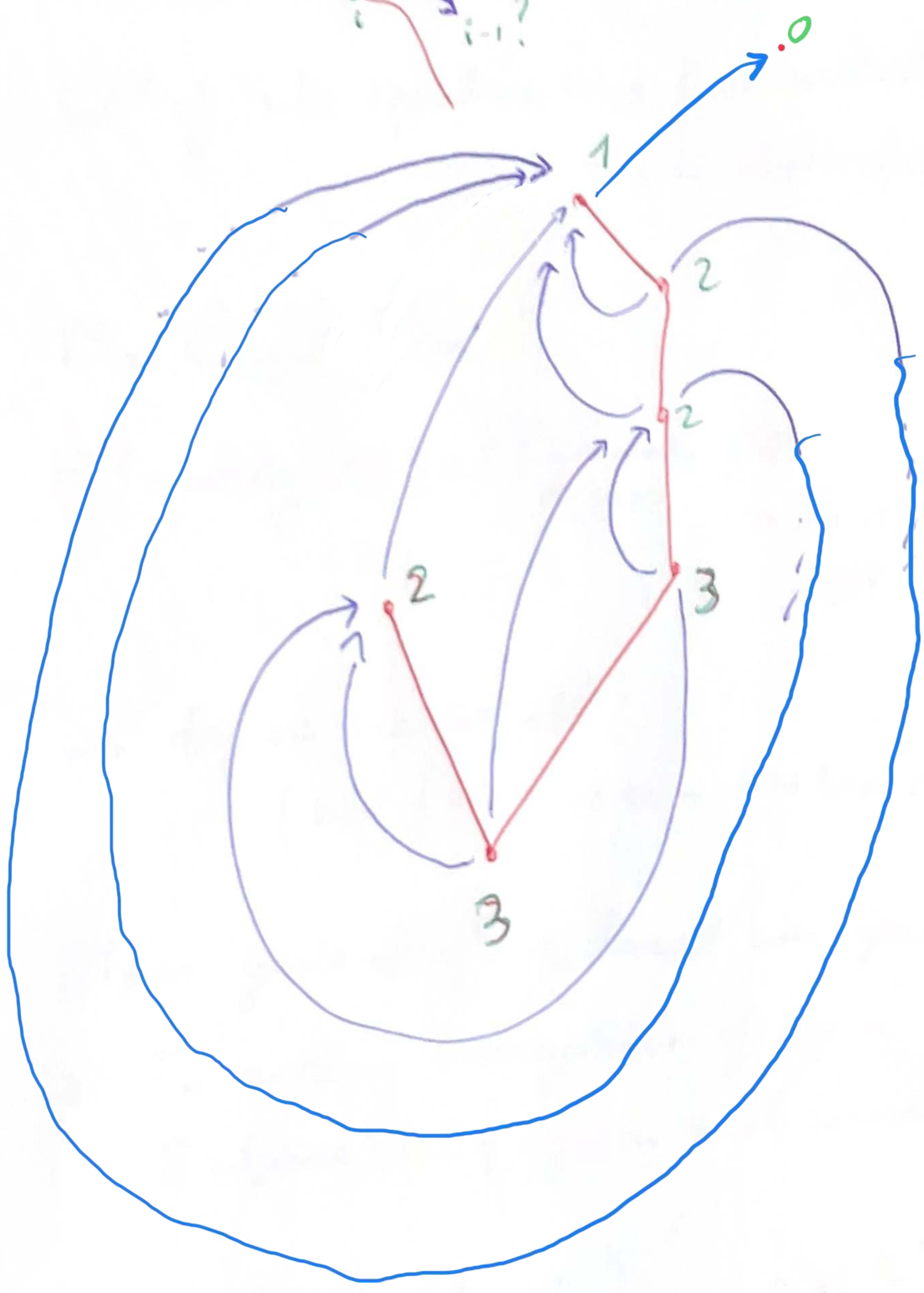


)

converse bijection:

- add missing "vertex 0".
- each corner i looks for its next i-1 around the tree:





IV Randomness:

(but highly non-trivial!)

Fundamental historical consequences. Starting points of intense research areas.

1/ Local behaviour

Sol° of Tutte equation ⇝ full control of root face degree (or root vertex, by duality).

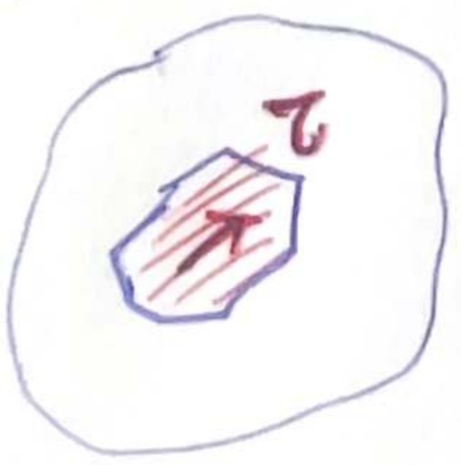
$M_n \in_{unif} \mathcal{M}_n$.

$\mathbb{P}(\text{root deg}(M_n)=k) \xrightarrow[n\to\infty]{} p_k$
↳ some explicit finite sequence with exponential tail
$p_k \sim c\, k^{+1/2}\, 2^{-k}$

→ degrees are small!
(kind of the same as in GW trees / plane trees).

More generally ⇝ local convergence.

Let T_n uniform triangulation of size n and τ (fixed) tri° of size m with boundary p.

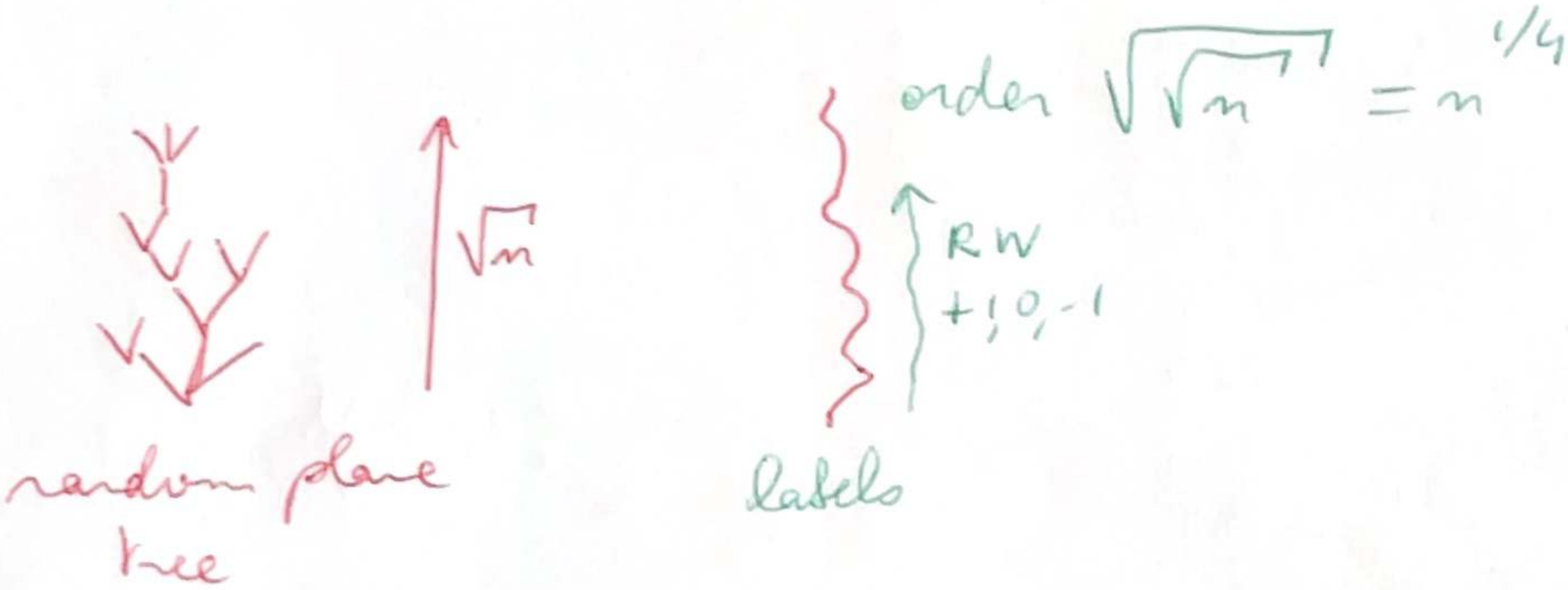
$\mathbb{P}$ (neighbourhood of root in T_n looks like τ: ) $=$ $\dfrac{\text{n of tri° (size } n-m\text{, perimeter } p)}{\text{n of tri° (size } n)}$ $\xrightarrow[]{n\to\infty}$ explicit nb a_p.

[uses analog of Tutte's eq for triangulations. See exercise]

Notion of local limit, Uniform Infinite Random Triangulation (UIPT, Angel & Schramm 2000's)

2/ Global behaviour

Schaeffer bijection $\Longrightarrow$ distances to a random vertex in random quadrangulation Q_n are distributed like **labels** in a random 1-Lipschitz labelled random tree



Fact: $\text{diameter}(Q_n) = O_p(n^{1/4})$ when $n \to \infty$

More generally

$$\frac{Q_n}{n^{1/4}} \xrightarrow[\text{some topology on metric spaces}]{n \to \infty} \text{Brownian map}$$ [Marckert-Mokkadem, Le Gall, Miermont, ...]

constructed by (difficult) continuous analogue of the Schaeffer bijection on continuous trees.

THE END

(thanks!).